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ON VARIATIONS IN THE SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS OF MIXED PROBLEMS OF ELASTICITY THEORY FOR DOMAIN VARIATIONS*

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The behaviour of the solution of the boundary value problem for a pseudo-differential equation (PDE), Green's function of this problem, and also some of their local and global characteristics, during variation of the domain is investigated. Formulas are proposed that enable the solution of a broad class of PDE in a domain to be expressed in terms of the solution in the near domain. Local characteristics of the solution are expressed in terms of the local characteristics of the solution in the near domain. A double asymptotic form of Green's function for both arguments tending to the domain boundary occurs in the variation formula. The variation of this double asymptotic form as the domain varies is expressed in terms of this same asymptotic form. The system of variation formulas obtained is closed. It enables the PDE solution in the domain to be reduced to the solution of an ordinary differential equation in functional space. The local characteristics of the solution can also be found by this method without calculating the solution itself. If there is sufficient symmetry in the initial operator, then conservation laws in the Noether sense are obtained for its Green's function and its asymptotic form. The behaviour of the quantities under investigation is studied under inversion.

The investigation of variations of the solutions of problems for the variation of the domain occurs in the paper by Hadamard /1/, who studied the variation in conformal mapping and obtained a formula similar to (1.4). The formula for the variation of the solution of the boundary value problem for an elliptic differential equation is obtained in /2/. Variation formulas for the case of the operator of the problem about a crack and a circular domain are obtained in /3, 4/. The Irwin formula /5/ is obtained from formulas (1.4) and (1.21) by substitution.

1. An important special case for which variation formulas will be obtained is the problem of a crack with the operator

$$\mathbf{A}u(x_1, x_2) = \Delta \int_S dy_1\,dy_2 \frac{u(y_1, y_2)}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}} \tag{1.1}$$

where S is the domain occupied by a plane normal separation crack in an unbounded body. It is known /3/ that the crack opening $u(x_1, x_2)$ is related to the density $f(x_1, x_2)$ of the normal discontinuing forces of the PDE

$$f(x_1, x_2) = 2(1-\nu^2)E^{-1}\mathbf{A}u(x_1, x_2) \tag{1.2}$$

with the boundary condition $u|_{\partial S} = 0$.

It is well-known that the solution of the PDE (1.2) has the asymptotic form $u(x_1, x_2) \sim CN(\xi)s^{1/2}$ as $(x_1, x_2) \to \xi \in \partial S$ if $(x_1, x_2) \in S$ moreover, $\mathbf{A}u(x_1, x_2) \sim N(\xi)s^{-1/2}$, where $(x_1, x_2) \notin S$. Here $s = \rho(x, \partial S)$, $N(\xi)$ is the stress intensity factor. For $(x_1, x_2) \notin S$ the function $\mathbf{A}u(x_1, x_2)$ describes the normal stress distribution on the continuation of the crack. Solutions of the PDE of the problem of a punch with a sharp tip /6/ have analogous properties.

In the general case let $A(x, y)$ be a generalized function in $\mathbf{R}^n \times \mathbf{R}^n$, and S a domain in $\mathbf{R}^n$. Let us consider the operator $\mathbf{A}$ in the domain S whose kernel is the function $A(x, y)$ bounded in $S \times S$, i.e.,

$$\mathbf{A}: f(x) \mapsto \mathbf{A}f(x) = \int_S A(x, y) f(y)\,dy$$

We assume that for a certain number $\alpha \in \mathbf{R}$ a unique solution of the equation $\mathbf{A}u = f$ exists for sufficiently smooth ∂S and f, where $u(x)$ has the asymptotic form $N'(x_0)s^{\alpha}/\Gamma(\alpha+1)$

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for a certain function N' as $x \to x_0 \in \partial S$, while $s = \rho(x, \partial S)$.

We introduce Green's function $G(x, y)$ and the influence function

$$E(x, y) = -\int_{\mathbb{R}^n} A(x, z) G(z, y)\, dz + \delta(x, y)$$

where $G(x, y)$ is assumed to be zero for $x \notin S$ or $y \notin S$. For the problem of a crack, the function $E(x, y)$ yields a normal stress at the point x on the crack continuation, and for a single concentration opening load applied at the point y the function $G(x, y)$ yields the opening of a crack at the point x under the same load.

Formulas expressing the variation of Green's function for the operators $\mathbf{A}$ that possess the following properties: $\mathbf{A}$ is a selfadjoint pseudodifferential operator, for sufficiently smooth ∂S

$$G(x, y) \sim \frac{g'(x_0, y)\, s^{\alpha}}{\Gamma(\alpha+1)}, \quad x \in S, \quad E(x, y) \sim \frac{e'(x_0, y)\, s^{-\alpha}}{\Gamma(-\alpha+1)}, \tag{1.3}$$
$$x \notin S$$
$$g'(x_0, y) = C(x_0, S)\, e'(x_0, y), \quad y \in S, \quad x_0 \in \partial S, \quad x \to x_0$$
$$s = \rho(x, \partial S)$$

will be presented here.

The formulas for the operator of the problem about a crack are derived in Sects.2–5. The notation without primes is used there and the coefficients in the asymptotic forms are not normalized to the Γ-function. In general, the proofs will be carried out without substantial changes.

We shall give the change in the domain as follows: let a family of domains S_t, $S_0 = S$ be given, where t is a parameter which is not necessarily real time. For the point $z \in \partial S$ let z_t be the point of intersection of ∂S_t with the perpendicular to ∂S_0 at the point z, and let $w(z)$ be the velocity of motion of z_t outside the domain. The function $w(z)$, $z \in \partial S$ will also characterize the rate of change of the domain. Let $G_t(x, y)$, $g_t'(z_t, y)$ and $e_t'(z_t', y)$, $x, y \in S_0$, $z \in \partial S_0$ denote the functions being investigated for the domain S_t.

In this case

$$\frac{\partial}{\partial t} G_t(x, y)|_{t=0} = \int_{\partial S} dz\, w(z)\, e'(z, y)\, g'(z, x) \tag{1.4}$$

and $e'(x, y) \sim \delta_{y_0}(x)\, s^{\alpha-1}/\Gamma(\alpha+1)$ for $y \in S$, $y \to y_0 \in \partial S$, where $e'(x, y)$ and $\delta_{y_0}(x)$ are considered as generalized functions on ∂S, $s = \rho(y, \partial S)$.

If the condition $E(x, y) \sim e^\circ(x, y_0)\, s^{\alpha}/\Gamma(\alpha+1)$ is satisfied as $y \to y_0 \in \partial S$, $y \in S$, then

$$\frac{\partial}{\partial t} E_t(x, y)|_{t=0} = \int_{\partial S} dz\, w(z)\, e^\circ(x, z)\, e'(z, y) \tag{1.5}$$

We assume that $A(x, y)$ depends only on $x - y$. We examine the next term in the asymptotic form for $e'(x - y)$:

$$e'(x, y) \sim \delta_{y_0}(x)\, s^{\alpha-1}/\Gamma(\alpha) + \varepsilon'(x, y_0)\, s^{\alpha}/\Gamma(\alpha+1)$$
$$y \to y_0 \in \partial S$$

In this case

$$\frac{\partial}{\partial t} g_t'(x_t, y)|_{t=0} = \int_{\partial S} dz\, (w(z) - (n_z, n_x)\, w(x))\, g'(z, y)\, \varepsilon'(z, x) - \tag{1.6}$$
$$(n_x, \partial/\partial y)\, g'(x, y)\, w(x)$$

and if the condition formulated before (1.5) is satisfied, then

$$\frac{\partial}{\partial t} e_t^\circ(x, y_t)|_{t=0} = \int_{\partial S} dz\, (w(z) - (n_z, n_y)\, w(y))\, e^\circ(x, z)\, \varepsilon'(z, y) - \tag{1.7}$$
$$(n_y, \partial/\partial x)\, e^\circ(x, y)\, w(y)$$

where n_q is the external normal to ∂S at the point $q \in \partial S$.

Moreover, all the subsequent terms of the asymptotic form $G(x, y)$ are evaluated in terms of $g'(x, y)$ and $\varepsilon'(x, y)$ as $x \to x_0 \in \partial S$. For instance, if

$$G(x, y) \sim g'(x_0, y)\, s^{\alpha}/\Gamma(\alpha+1) + g_{(1)}'(x_0, y)\, s^{\alpha+1}/\Gamma(\alpha+2) \tag{1.8}$$
$$x \to x_0 \in \partial S$$

where $s = \rho(x, \partial S)$, then

$$g_{(1)}'(x, y) = \int_{\partial S} dz\,(n_z, n_x)\, g'(z, y)\, \varepsilon'(z, x) + (n_x, \partial/\partial y)\, g'(x, y)\, w(x) \tag{1.9}$$

and for the vector l tangent to ∂S at x

$$\left(l, \frac{\partial}{\partial x}\right) g'(x, y) + \left(l, \frac{\partial}{\partial y}\right) g'(x, y) + \int_{\partial S} dz\,(l, n_z)\, g'(z, y)\, \varepsilon'(z, x) = 0 \tag{1.10}$$

If $A(x, y)$ depends only on the distance between x and y, then in addition to (1.10), several conservations laws are still satisfied in the Noether sense. Let $a \perp (x - y)$, and for $n \geqslant 3$ let $b \perp a, (x - y)$.
Then

$$\int_{\partial S} dz\,((z - x, x - y)(n_z, a) - (z - x, a)(n_z, x - y)) \times \tag{1.11}$$
$$g'(z, y)\, \varepsilon'(z, x) + |x - y|^2 (a, \partial/\partial y)\, g'(x, y) = 0$$

$$\int_{\partial S} dz\,((z - y, b)(n_z, a) - (z - y, a)(n_z, b))\, g'(z, y)\, \varepsilon'(z, x) = 0 \tag{1.12}$$

If $A(x, y)$ is a homogeneous function of $x - y$ of degree of homogeneity β, then $\beta = -n - 2\alpha$ and

$$(2n + \beta)\, g'(x, y) + \left(y - x, \frac{\partial}{\partial y}\right) g'(x, y) + \int_{\partial S} dz\,(z - x, n_z)\, g'(z, y)\, \varepsilon'(z, x) = 0 \tag{1.13}$$

For an arbitrary function $A(x, y)$ satisfying the assumptions (1.3), a closed formula is not obtained successfully for the variation, and only the following relationship can be written:

$$\frac{\partial}{\partial t} g_t'(x_t, y)|_{t=0} + g_{(1)}(x, y)\, w(x) = \int_{\partial S} dz\, w(z)\, g'(z, y)\, \varepsilon'(z, x) \tag{1.14}$$

If $A(x, y)$ satisfies the relationships and the homogeneity, and the invariance relative to rotation, then it has the form $|x - y|^\beta$ and still possesses certain properties of invariance relative to inversion. More accurately, for an inversion with respect to a circle with centre at zero and a radius R, if x_i denotes the image of the point x and S_i of the domain S, then

$$G_S(x, y) = R^{2n-4\alpha} |x|^{2\alpha-n} |y|^{2\alpha-n} G_{S_i}(x_i, y_i) \tag{1.15}$$

$$g_S'(x, y) = R^{2n-2\alpha} |x|^{-n} |y|^{2\alpha-n} g_{S_i}'(x_i, y_i) \tag{1.16}$$

where the subscripts on the functions G and g' denote the domains for which they are evaluated. In this case the following conservation laws hold for any vector a:

$$\begin{aligned}&(2(y - x, a)(y - x) - |y - x|^2 a, \partial/\partial y)\, g'(x, y) + \\ &(2n + \beta)(y - x, a)\, g'(x, y) = \\ &\int_{\partial S} dz\,(2(z - x, a)(z - x, n_z) - |z - x|^2 (a, n_z))\, g'(z, y)\, \varepsilon'(z, x)\end{aligned} \tag{1.17}$$

In all $(n + 1)(n + 2)/2$ identities (inclusing (1.14)), which equals the dimensionality of the conformal group of n-space, are obtained for the function g. Moreover, a variation formula for ε' can be obtained for such an operator A, which we write for $n = 2$ for convenience

$$\begin{aligned}\frac{\partial}{\partial t} \varepsilon_t'(x_t, y_t)|_{t=0} = &\int_{\partial S} dz\,[w(z) - w(x)(n_z, n_x) - (x - y, n_x w(x) - \\ &n_y w(y))(z - x, n_z)|x - y|^{-2} - (x - y, l_x w(x) - l_y w(y)) \times \\ &(z - x, l_z)|x - y|^{-2}]\, \varepsilon'(z, x)\, \varepsilon'(z, y) - n(x - y, n_x w(x) - \\ &n_y w(y))|x - y|^{-2} \varepsilon'(x, y)\end{aligned} \tag{1.18}$$

(for $n > 2$ the third component in the square brackets will have a more complex form). Here l_q is a unit vector tangent to ∂S at the point q. Only the transformation properties of $\varepsilon'(x, y)$ for parallel transfers, rotations, and extensions are used in this formula. For $w(z)$ of the form (z, n_z), (z, l_z), (a, n_z), where $a \in R^2$, four conservation laws, analogous to (1.10)–(1.13), are obtained from (1.18) for ε' (two of them will be linearly independent). Using the identity

$$\varepsilon_S'(x, y) = R^{2n} |x|^{-n} |y|^{-n} \varepsilon_{S_i}'(x_i, y_i) - \alpha(n_y, y)|y|^{-2} \delta(x, y) \tag{1.19}$$

which is analogous to (1.15) and (1.16), conservation laws for ε' can be written that are analogous to (1.17) for g'.

The formulas

$$\frac{\partial}{\partial t} u(x)|_{t=0} = \int\limits_{\partial S} dz\, N'(f, z)\, w(z)\, g(z, x) \tag{1.20}$$

$$N'(f, z) = \int\limits_{S} dx\, e'(z, x)\, f(x) \tag{1.21}$$

$$\frac{\partial}{\partial t} N'(f, x)|_{t=0} = \int\limits_{\partial S} dz\, (w(z) - (n_z, n_x)\, w(x))\, N'(f, z)\, \varepsilon'(z, x) + N'\left(\left(n_x, \frac{\partial}{\partial y}\right) f(y), x\right) w(x) \tag{1.22}$$

are valid for variations of the solution of the equation $Au = f$ (A satisfies (1.3)).

Here N' is the local characteristic of the solution u on ∂S $u(x) \sim N'(f, x_0)\, s^{\alpha}/\Gamma(\alpha+1)$, $s = \rho(x, \partial S)$, $x \to x_0 \in \partial S$. Formula (1.22) holds if $A(x, y)$ depends only on $x - y$.

These formulas can be used to evaluate $N'(f, x)$ for a polynomial or lumped load $f(x)$ or a load $f(x)$ in the form of an exponential polynomial in an arbitrary domain S if the solution of this problem is known in some, say circular, domain S' as is also the function $\varepsilon'(x, y)$ for this domain. Indeed, we include the domains S and S' in the family of domains S_t with $S_0 = S'$, $S_1 = S$. Formulas (1.18) and (1.6) yield a system of two ordinary first-order differential equations for $\varepsilon_t'(x, y)$ and $e'(x, y)$, while (1.18) and (1.22) yield the very same for $\varepsilon_t'(x, y)$ and $N'(\exp(a, \cdot), x)$.

For a polynomial load $P(x)$ formulas (1.18) and (1.22) yield a system of $k+1$ ordinary first-order differential equations for $\varepsilon_t'(x, y)$ and $N'(P_i, x), i = 1, \ldots, k$, where $P_1 = P$ and $P_2, \ldots, P_k$ are all non-zero partial derivatives of the polynomial P. Note that for $A(x, y) = |x - y|^{\beta}$ the derivatives $g'(x, y)$ with respect to y in (1.6) can be eliminated by using (1.10), (1.11) and (1.13).

By using (1.20) even the solution u of the PDE $Au = f$ can be evaluated analogously.

We assume that a dependence of the form

$$w(x) = F(N(P, x)) \tag{1.23}$$

for the growth rate of the crack at the point x on the stress intensity factor at x exists for the given material under the conditions of the problem of a crack. In this case (1.23), (1.21), (1.6) and (1.18) can be understood as the equations for the evolution of the crack with time, and similarly (1.23), (1.22), (1.18) for the polynomial load.

For domains of the "half-plane" and "circle" form and the operator of the problem about a crack, formulas are obtained for $e'(x, y)$ and $\varepsilon'(x, y)$ from the formulas in Sect.6 taking accout of the relationships

$$e' = \sqrt{\pi} e, \quad \varepsilon' = \varepsilon\pi\, 2$$

2. We will prove (1.4) for the Green's function variations. For simplicity, the proof will be carried out for the case of the operator (1.1) of the problem of a crack.

The crux of the method is the utilization of a variable curvilinear coordinate system in which ∂S is described by an equation independent of t. In this coordinate system, the operator depends on t and application of the usual perturbation theory in the form $\mathbf{X}_{n+1} = \mathbf{X}_n - \mathbf{X}_n (\mathbf{B} - \mathbf{X}_n^{-1}) \mathbf{X}_n$ is possible, where $\mathbf{X}_k$ are the next approximations to the operator $\mathbf{B}^{-1}$. If $\mathbf{X}_n$ has a first order of closeness to $\mathbf{B}^{-1}$, then $\mathbf{X}_{n+1}$ will have a second order of closeness. Since there is a small factor in the second term, then $\mathbf{Z} = \mathbf{X} - \mathbf{Y}(\mathbf{B} - \mathbf{X}^{-1})\mathbf{X}$ has a second order of closeness to $\mathbf{B}^{-1}$ when $\mathbf{X}$ and $\mathbf{Y}$ have a first order of closeness.

Let $v(x)$ be a vector field in a plane, and let $h_t(x)$ be the solution of the equation $\partial h_t(x)/\partial t = v(h_t(x))$, $h_0(x) = x$. Let $q_t = h_t(q)$. We determine the action of h_t on the function and on the density by means of the formulas $h_{t*} f(x) = f(h_{-t}(x))$, $h_{t*}(f(x)\, dx) = f(h_{-t}(x))\, d(h_{-t}(x))$. By the action of two variables on the function the superscript on the h_{t*} will denote the variable on which h_{t*} acts.

We take $h_{t*}^{x} h_{t*}^{y} G(x, y)\, dy$ as $\mathbf{X}$ (for convenience we replace $G(x, y)$ by the density $G(x, y)\, dy$). We take $G(x, y)\, dy$ as $\mathbf{Y}$. The operator $\mathbf{B}_t$ will be the limitation of the operator $\mathbf{A}$ in the domain S_t. Then

$$h_{-t*}^{y} Z_t(x, y) = h_{t*}^{x} G(x, y) - \int G(x, z)\, dz \left(\int A(z, q)\, dq\, h_{t*}^{q} G(q, y) - \delta_{zy} + h_{t*}^{z} E(z, y)\right) = h_{t*}^{x} G(x, y) - \int dq\, \delta_{xq} h_{t*}^{q} G(q, y) + \int dq\, E(q, x)\, h_{t*}^{q} G(q, y) + G(x, y) - \int dz\, G(x, z)\, h_{t*}^{z} E(z, y)$$

It can be assumed that $v(x) = v(y) = 0$. Then

$$G_t(x, y) = Z_t(x, y) + o(t) = G(x, y) + \int dq\, E(q, x)\, h_{t*}^{q} G(q, y) - \int dz\, h_{t*}^{z} E(z, y)\, G(x, z) + o(t)$$

The integrands are non-zero only in a small neighbourhood of ∂S. We introduce the coordinates l and n, where l is the base of a perpendicular dropped from a point to ∂S and n is the coordinate of a point on this perpendicular measured outside from ∂S. Since the function $w(x)$ from Sect.1 is in agreement with the n-component of the field $v(x), x \in \partial S$, the first integral is

$$\int\limits_{l\in\partial S,\, w(l)\geq 0} dl \int\limits_0^{tw(l)+o(t)} dn\,(e(l,x)\,n^{-1/2}+o(n^{-1/2}))\,(g(l,x)\,(tw(l)-n)^{1/2})+ o((tw(l)-n)^{1/2}) = t\frac{\pi}{2}\int\limits_{l\in\partial S,\, w(l)\geq 0} dl e(l,x)\,g(l,y)+o(t),\quad t\geq 0$$

Evaluating the second integral analogously, we obtain (1.4). Formula (1.5) is obtained by applying the operator A to both sides of (1.4).

3. Differentiating the asymptotic form (1.8) of the function $G(x,y)$ as $x\to x_t\in\partial S_t$, we obtain

$$\frac{\partial}{\partial t}G_t(x,y)|_{t=0}=\frac{1}{2}w(x)\,g(x,y)\,s^{-1/2}+\left(\frac{\partial}{\partial t}g_t(x_t,y)|_{t=0}+\frac{3}{2}w(x_0)\,g_{(1)}(x_0,y)\right)s^{1/2}+ds^{1/2}\Big)$$

Comparing with (1.4), we obtain that $e(s,x)=\pi^{-1}\delta_{x_0}(s)+O(s^{1/2})$ and

$$\frac{\partial}{\partial t}g_t(x_t,y)|_{t=0}=\frac{\pi}{2}\int\limits_{\partial S}dz w(z)\,g(z,y)\,\varepsilon(z,x)-\frac{3}{2}g_{(1)}(x,y)\,w(x) \tag{3.1}$$

where ε is the coefficient of $s^{1/2}$ in the asymptotic form of the function e.

If the family of domains S_t is obtained from S by translation of the normal ∂S to the vector n_x at the point x, then $w(s)=(n_s,n_x)$. If meanwhile $A(s,y)$ depends only on $s-y$, then $g_t(x_t,y)=g(x,y-tn_x)$. Comparing theis expresssion with (3.1), we obtain (1.14), i.e.,

$$-\left(n_x,\frac{\partial}{\partial y}\right)g(x,y)=\frac{\pi}{2}\int\limits_{\partial S}dz\,(n_z,n_x)\,g(z,y)\,\varepsilon(z,x)-\frac{3}{2}g_{(1)}(x,y)$$

We deduce (1.6) from the last two formulas, and formula (1.7) is also derived analogously.

4. If the operator is invariant under a transformation of the plane Φ, then $G_{\Phi S}(\Phi(x),\Phi(y))=G_S(x,y)$, etc. Consequently, if the operator A is invariant under the group of diffeomorphisms h_t generated by the vector field v, then $g_t(x_t,y)=g(h_{-t}(x_t),h_{-t}(y))$, where $S_t=h_tS$. Hence $\partial g_t(x_t,y)/\partial t$ is calculated and by comparing this expression with (1.6), we obtain a conservation law for g. Analogous reasoning can be performed even in the case when the operator is multiplied by a constant in the transformation of the plane Φ. From invariance with respect to the field $v(s)=a$, (1.10) is obtained for $a\perp n_x$ (the field v generates a parallel translation) with respect to the fields $v(s)=(z-x,x-y)\,a-(z-x,a)(x-y)$ and $v(z)=(z-x,a)\,b-(z-x,b)\,a$, where $b\perp a$, $x-y$ and $a\perp x-y$, (1.11) and (1.12) are obtained (the fields v generate rotation about the point x) and (1.13) is obtained with respect to the field $v(z)=z-x$ (the field v generates homothety relative to x

To prove relationships (1.15), (1.16), (1.19), we note that from the similarity of the triangles OXY and OX_iY_i it follows that $dx_i=|x|^{-2n}R^{-2n}dx$, $|x_i-y_i|=|x-y|\,R^2\cdot|x|^{-1}|y|^{-1}$. Consequently, (1.16) and (1.19) follow from (1.15) without difficulty. To prove (1.15) it is sufficient to prove that if one acts on the right side with the operator A, we obtain a δ-function at the point y in the domain. But

$$\int dz\,dy A(x,y)\,G(y,z)\,\varphi(z)=\int dz\,dy A(x_i,y_i)\cdot R^{-2\beta}|x|^{\beta}|y|^{\beta}G_{S_i}(y_i,z_i)\times R^{2n+2\beta}|y|^{-2n-\beta}|x|^{-2n-\beta}\varphi(z)=\int dz_i\,dy_i|x_i|^{-\beta}|z_i|^{\beta}A(x_i,y_i)\,G(y_i,z_i)\,\varphi(z_{ii})$$

and $\operatorname{Supp}\varphi(z_i)|z|^{\beta}_i\subset S_i$ if $\operatorname{Supp}\varphi\subset S$, consequently, the last integral is $|x_i|^{-\beta}\varphi(x_{ii})|x_i|^{\beta}=\varphi(x)$.

We note that the component with the δ-function in (1.19) is obtained from the first term in the asymptotic form of the function e, taking the non-linearity of the substitution $\rho(y,\partial S)\mapsto\rho(y_i,\partial S_i)$ into account. Formula (1.17) is obtained because of the covariance of (1.1) relative to the vector field $v(z)=2(z-x,a)(z-x)-(z-x)^2a$, which generates the composition of an inversion with reflection.

5. To prove (1.18) we note that for $w(x)=w(y)$ it simplifies to

$$\frac{\partial}{\partial t}\varepsilon_t(x_t,y_t)=\frac{\pi}{2}\int dz w(z)\,\varepsilon(s,x)\,\varepsilon(z,y)$$

In this form it follows directly from (1.6) since g and e are proportional.

In the general case a combination of shifts, rotations, and extensions φ_t can be selected such that $\varphi_t(x)=x_t$, $\varphi_t(y)=y_t$.

The vector field corresponding to φ has the form

$$w(x)\, n_x + (w(y)\, n_y - w(x)\, n_x,\ y - x)\, |x - y|^{-2} (y - x) + (w(y)\, n_y - w(x)\, n_x,\ a)\, |a|^{-2} a$$

where $a \neq 0,\ a \perp x - y$. Since A is only multiplied by a constant for the transformation φ then

$$\varepsilon_{\varphi_t S}(\varphi_t(x), \varphi_t(y)) = \exp(-2(w(y)\, n_y - w(x)\, n_x,\ y - x)\, |x - y|^{-2} t + o(t))\, \varepsilon_S(x, y)$$

The condition $w(x) = w(y) = 0$ is satisfied for the family of domains $\varphi_{-t} S_t$, consequently (1.18) follows from the deduced transformation formulas for ε.

Formula (1.20) is derived in the same way as (1.4), and (1.22) in the same way as (1.6).

6. The function $E(x, y)$ is known for the domain S in the form of a half-plane and a circle. For the half-plane /7/

$$E(x, y) = \pi^{-2} |x - y|^{-2} \rho(x, \partial S)^{-1/2} \rho(y, \partial S)^{1/2}$$
$$e(x, y) = \pi^{-2} |x - y|^{-2} \rho(y, \partial S)^{1/2}$$

The asymptotic formula for $e(x, y)$ as $y \to \partial S$ is easily verified. Moreover, $\varepsilon(x, y) = \pi^{-2} |x - y|^{-2}$. Hence, we obtain the following formula for a circular domain of radius R

$$e(x, y) = \pi^{-2} |x - y|^{-2} [s(2R - s)]^{1/2} (2R)^{-1/2} \tag{6.1}$$

derived by another method in /6/. Here $s = \rho(y, \partial S)$. In addition

$$\varepsilon(x, y) = \pi^{-2} |x - y|^{-2} - (2\pi R)^{-1} \delta_y(x) \tag{6.2}$$

We will calculate the stress intensity factor for an elliptical crack under constant load to a first approximation. The second component in (1.22) vanishes in this case. For a circular crack we obtain $N(1, x) = \sqrt{2R}/\pi$ for (1.21) and (6.1).

For $w(x) = -\sin^2\varphi$, where φ is the angular coordinate of the point x on the circle, the circle will be deformed into an ellipse with $a = R,\ b = R - t$. We introduce the standard angular coordinate β on the ellipse according to the rule $x_1 = a\cos\beta,\ x_2 = b\sin\beta$. In this case $\beta(x_t) = \varphi(x) + O(t)$. Since $N(1, x)$ is independent of x for $t = 0$, the derivatives of N with respect to t are in agreement for $\beta = \text{const}$ and $\varphi = \text{const}$. After calculations, we obtain

$$\frac{\partial}{\partial(a - b)}\Big|_{a = \text{const}} N_{\text{ell}}(1,\ x) = -\frac{\cos^2\varphi}{\pi\sqrt{2R}}$$

From the expressions with such a derivative and with the necessary value for $a = b$ we select an expression symmetric in a and b (such a selection will certainly influence the magnitude of the remainder term). The most natural formula is the following

$$N_{\text{ell}}(1, \varphi) = \sqrt{2}\sqrt{b\cos^2\varphi + a\sin^2\varphi}/\pi + O((a - b)^2 a^{-3/2}) \tag{6.3}$$

This formula is in complete agreement with the exact formula presented in /8/, say (β is the standard coordinate on the ellipse mentioned above)

$$N_{\text{ell}}(1, \beta) = (\sin^2\beta + \cos^2\beta \cdot b^2/a^2)^{1/4} \sqrt{b}/\sqrt{2}\,\mathrm{E}(k),\quad k^2 = 1 - b^2/a^2$$

The magnitude of the remainder term in (6.3) can be demonstrated as follows: if as is assumed in (1.22), φ is understood to be the angle between the major semi-axis and the direction from the centre to the point x, then (6.3) yields identical relative errors for $\varphi = 0$ and $\varphi = \pi/2$, i.e., at the vertices of the ellipse (therefore, it yields the correct ratio between the magnitudes of the intensity factors at these points), where the remainder term is less than 9% at the vertices for $b/a = 0.5$, about 27% for $b/a = 0.3$, less than 2% at a point with coordinate $\varphi = \pi/4$ for $b/a = 0.5$, and 4.5% for $b/a = 0.3$.

In conclusion we note that it follows from (1.18) and (6.2) that for any domain S as $x \to y$

$$\varepsilon(x, y) = \pi^{-2} |x - y|^{-2} - (2\pi)^{-1} k_x \delta_x(y) + O(1)$$

where k_x is the curvature of ∂S at the point x.

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DETERMINATION OF THE AVERAGE CHARACTERISTICS OF ELASTIC FRAMEWORKS*

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A method is proposed for the approximate calculation of the average elastic characteristics of fine-celled framework structures of periodic configuration. The method is based on approximation of the "cell problem" of the theory of averaging /1—4/ by problems on the deformation of appropriate structures of beam, shell, etc., types. It is shown that the approximate values obtainable for the average characteristics and the solution of their appropriate problems are distinguished from the exact solutions by a quantity determined only by the error of the model being used. Examples are considered, namely, beam and box frameworks, and the construction of a framework with negative Poisson's ratios.

Methods for the average description of bodies containing a large number of fine vacancies /1, 2/ enable the structure of perodic configuration to be replaced by the consideration of continuous bodies similar in mechanical behaviour but with so-called average characteristics. The problem of finding the average characteristics is reduced in /2/ to the so-called cell problem of elasticity theory whose solution is quite difficult. At the same time, the solution of the cell problem in framework structures whose periodic element is a beam- or shell-type structure can be obtained by approximate methods to any accuracy, which is governed merely by the selection of the model.

An elastic structure of periodic configuration with perodicitiyy cell (PC) in the form of a parallelepiped $P_\varepsilon = \varepsilon P_1 = \{\varepsilon x : x \in P_1\}$ is considered, where $P_1 = \{x \in R^n: -\mu_i/2 \leqslant x_i \leqslant \mu_i/2, i = 1, \ldots, n\}$ $(n = 2, 3)$ is a rectangular parallelepiped with a characteristic length of the sides equal to one $(\mu_i \sim 1)$. The elastic material does not occupy the whole PC P_ε but only a part K_ε, which can be represented in the form $K_\varepsilon = \varepsilon K_1$. Under the condition that the characteristic (absolute or relative) PC dimension $\varepsilon \to 0$, production of the average is possible /2/. To determine the average elastic constants $\{\bar{a}_{ijkl}\}$ of a medium formed on the basis of the PC P_ε the part K_ε occupied by a material with the elastic constants $\{a_{ijkl}\}$ should minimize the functional /2/

$$F(\mathbf{u}) = \frac{1}{\operatorname{mes} P_1} \int_{K_1} a_{ijkl} (\operatorname{def} \mathbf{u})_{ij} (\operatorname{def} \mathbf{u})_{kl} \, dx \quad (1)$$

$$(\operatorname{def} \mathbf{u})_{ij} = {}^1/_2 (u_{i,j} + u_{j,i})$$

in the set of functions $\{H_2^1(P_1)\}^n$ under the additional conditions

$$\int_{P_1} \mathbf{u}(\mathbf{x}) \, d\mathbf{x} = 0 \quad (2)$$

$$\mathbf{u} - {}^1/_2 (x_\alpha \mathbf{e}_\beta + x_\beta \mathbf{e}_\alpha) \in \Pi_1 \quad (3)$$

Here and henceforth, Π_1 is a class of functions periodic in P_1 ($\mathbf{e}_\alpha, \mathbf{e}_\beta$ are basis unit

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